Signal Priority in a Multi-Asset World

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December 1, 2022

Abstract

I study a competitive equilibrium populated by a generalist and two specialists. There are two Lucas trees with output correlation, and two signals on the conditional growth rate of each tree. While the generalist uses two signals to infer the conditional mean of both trees, specialists each use *one* signal, albeit with higher precision, to learn about *both* trees – since output is correlated. I find that when precision is low, the generalist eventually vanishes from the market due to an inability to hold decisive trade positions; as the generalist is always sandwiched between two aggressive specialists, it is not until precision is high that the information advantage of knowing many things pays off.

A fox knows many things, but a hedgehog knows one big thing

–Isaiah Berlin on Archilochus

1 Introduction

I study a competitive equilibrium populated by a generalist and two specialists, cast as foxes and hedgehogs, in a difference-of-opinion framework. Agents maximize expected utility but must form beliefs about time-varying expected returns. There are two Lucas trees in this economy with output correlation, and two signals on the conditional growth rate of each tree. While the generalist uses two signals to infer the conditional mean of both trees, specialists each use *one* signal, albeit with higher ability/skill/precision, to learn about *both* trees –

since output is correlated. I study the long-run survival of each agent in this economy and characterize regions for different values of skill in which one group dominates the other.

I find that if the precision/ability of the generalist is low, the generalist is eventually driven out of the market. At intermediate levels of precision, dominance depends on the separation specialists have over the generalist. When the generalist's precision is high−so that there is no room for more separation−, the generalist strictly dominates. The primary mechanism behind these results is whether the generalist is able to secure decisive gains. Because the generalist observes both signals at lower precision, his trade positions are always in-between the two specialists. If the generalist believes that signals are noisy, more information is harmful because it prevents the generalist from taking uninformed positions that could work to his favor, which in noisy environments turn out to be important for securing meaning trade gains.

My results speak to the innocuous question: is it better to know many things or one thing really well; in financial markets, the question is already well studied. This paper departs however by first assuming the *co-existence* of both generalist and specialist − the first to my knowledge to do so, as opposed to the single rational agent models common in the optimal inattention literature.

Further in contrast, I study optimality implicitly through survival. According to the market selection hypothesis of Alchian (1950), Friedman (1953), and later Fama (1965), rational expectations is justified because irrational traders − or agents with incorrect beliefs to be precise, should eventually lose all of their wealth and vanish in the long run; in consequence, prices converge to the expectations of the most rational agents[1]. Sandroni (2000) formalizes this theory and shows that the market indeed favors agents who consistently make the best predictions without the need to assume rationality. Adding on, Blume and Easley (2006) show that if utility is time-additive and markets are complete, it is a result of the first welfare theorem that an agent receives more consumption in the states that he believes are more likely to occur. Hence, the survival of a generalist/specialist in the presence of the other directly implies who consistently makes the most accurate predictions, and I quantify the expected long-run share of consumption across different environments.

[1]See the introduction in Blume and Easley (2006) for an exposition on this subject

Does co-existence matter? In short, yes if only to overturn a conclusion in Admati and Pfleiderer (1987). In their paper, the authors study an example with two risky assets and two signals in which signal errors are correlated. They show that if prices are uninformative, signals are complements for low and high values of precision and substitutes for intermediate values. In contrast, my paper shows that it is strictly better to be a specialist at low levels of precision and in intermediate values, how much more the specialist knows over the generalist matters substantially.

Finally, I explore a recurring theme in the literature that generalists are better equipped to handle recessions[2]. My model explains this intuition by the same logic – it's not necessarily that generalists can time the market, but that they never commit to a single position. I show that even if the generalist is expected to be driven out, if the sample path is characterized by extreme events, the generalist survives because of his innate buffer against large swings in his consumption profile.

1.1 Related Literature

This paper builds on the literature on disagreement. Scheinkman and Xiong (2003) and Basak (2005) are two early papers. In Dumas et al. (2009), an irrational group of agents mistakenly believe that a signal is informative. As a result, irrational investors create their own noise à la De Long et al. (1990). In turn, the rational agents who are risk-averse are willing to accept lower expected returns given the presence of excess volatility. In the very long run, irrational agents eventually vanish from the market. In contrast, Borovička (2020) shows that irrational agents with recursive utility can dominate the equilibrium in the long run because the precautionary savings motive allows irrational agents to survive.

On information specialization, Goldstein and Yang (2015) show that it is optimal to specialize in subsets of news categories if there are large costs to processing different signals. Massa et al. (2020) follow the trades of institutional investors and find that investors with a high degree of news specialization outperform investors with a low degree of news specialization on average; in particular, "When specialists have larger buy positions on a stock than generalists, the stock will have higher future returns." Hameed et al. (2015) find evidence

[2]See for example Kacperczyk et al. (2016) and Zambrana and Zapatero (2021)

that when analysts revise the earnings of a firm that has high coverage, it changes the stock prices of similar firms significantly. Furthermore, they find evidence that analysts cover that firm in the first place because its fundamentals correlate most strongly with other firms in the same sector. Supporting substitutability of information, the authors argue that investors use information on one stock to price other stocks. Similarly, Mondria (2010) shows that if an investor has limited capacity to process information, the investor chooses to specialize in the signal that is potentially informative about many assets.

Finally, on market timing, Kacperczyk et al. (2014) find evidence that mutual fund managers who do well in expansions are also able to time the market in recessions – in other words, this particular skill is symmetric. Kacperczyk et al. (2016) show that one value mutual fund managers provide is performance during business cycles. In their model of optimal inattention with signals at the market and firm level, the volatility of the market spikes during recessions so it is optimal for the agent to allocate towards signals about the market (a generalist in this framework is a manager who learns about the market as opposed to someone who learns many things). Massa et al. (2020) find evidence that generalists switch to market related information during bear markets while specialists do not shift outside of their category. Lastly, Zambrana and Zapatero (2021) provide evidence that mutual funds outperform when they assign market timers to be generalists and superior stock pickers to be specialists, possibly a result of better matching.

2 Model

2.1 Endowment Economy

There are two Lucas trees, A and B. The cumulative cash flow process in the economy can be described by the stochastic differential equations:

$$\frac{\mathbf{d}X_t^i}{X_t^i} = \mu_t^i \mathbf{d}t + \sigma \left(\sqrt{\frac{1+\sqrt{1-\rho^2}}{2}} \mathbf{d}Z_t^i + \sqrt{\frac{1-\sqrt{1-\rho^2}}{2}} \mathbf{d}Z_t^j \right) \tag{1}$$

$$i \neq j; \ \ i,j \in \{A, B\}$$

One way to think of (1) is that it describes the real GDP between two partially integrated countries in which a temporary productivity shock to country i spills over to the productivity of country j. The growth rate of endowment μ_t^i is mean-reverting,

$$\mathbf{d}\mu_t^i = -\zeta(\mu_t^i - \bar{\mu})\mathbf{d}t + \sigma_\mu \mathbf{d}B_t^i \tag{2}$$

where $\zeta > 0$ is the speed of mean reversion, $\bar{\mu}$ is the long run rate of growth, $\mathbf{d}B_t^i$ are shocks to the growth rate and $\mathbf{d}Z_t^i$ are shocks to output. σ_μ is the diffusion for the rate of growth and it is less than σ, the volatility of output. Importantly, ρ is the correlation in output, controlling the weight that each output shock contributes to the output of a Lucas tree.

Furthermore, there are two public signals that are perfectly informative about shocks to the growth rate,

$$\mathbf{d}s_t^i = \mathbf{d}B_t^i$$

and agents are endowed with skill/ability to read the signals. One can see that just by observing $\mathbf{d}s_t^i$, an agent can form a precise estimate of $\mathbf{d}\mu_t^i$ and can consequently back out the sum of the shocks $\mathbf{d}Z_t^i, \mathbf{d}Z_t^j$ from output, $\mathbf{d}X_t^i/X_t^i$. By knowing the total output shock and accounting for the weights that each Brownian motion contributes, the agent can then reasonably back out $\mathbf{d}\mu_t^j$ from $\mathbf{d}X_t^j/X_t^j$ without having to "study" asset j. In this manner, an agent can use information on just one asset to price both in the spirit of Hameed et al. (2015).

2.2 Agents and Information Sets

There are three agents: 1 generalist and two specialists. Agents only observe realized output but know the parameter values for $\{\rho, \bar{\mu}, \zeta, \sigma, \sigma_\mu\}$. As I will expand on later, agents in this economy trade financial securities in each period and must continuously estimate μ_t^i from the history of outputs to balance their portfolio. Thus, uncertainty over μ_t^i propels the dynamics of this paper.

To make things interesting, no agent in this economy can perfectly read the signals. An appropriate interpretation could be that all the information to price securities is readily available but individuals each face constraints in capacity or ability.

In particular, the generalist believes that the dynamics of the signals obey

$$\mathbf{d}s_t^A = \phi' \mathbf{d}B_t^A + \sqrt{1-(\phi')^2}\mathbf{d}W_t^A$$
$$\mathbf{d}s_t^B = \phi' \mathbf{d}B_t^B + \sqrt{1-(\phi')^2}\mathbf{d}W_t^B$$

In other words, the generalist believes that the signal is a weighted average of two Brownian motions in which one Brownian motion is informative about the fundamental and the other is just pure noise. We can interpret $0 \leq \phi' \leq 1$ as his endowed ability or skill to read the signal while simultaneously functioning as the signal precision. To hammer the point, when ϕ' is low the generalist *believes* that the signal is mostly noise driven by $\mathbf{d}W_t^i$.

On the other hand, specialist 1 (who specializes in asset A) believes that the signals obey

$$\mathbf{d}s_t^A = \phi \mathbf{d}B_t^A + \sqrt{1-(\phi)^2}\mathbf{d}W_t^A$$
$$\mathbf{d}s_t^B = \mathbf{d}W_t^B$$

in which $\phi > \phi'$ since specialist 1 knows one big thing really well. In comparison to the generalist, specialist 1 correctly assigns a greater weight to the informativeness of signal $\mathbf{d}s_t^A$, but because he cannot "read" $\mathbf{d}s_t^B$, he treats the second signal as pure noise. The same scenario holds symmetrically for specialist 2 who specializes in asset B. Thus, this set-up tries to match as closely as possible the environment in which we can study the question: is it better to know many things or one thing really well?

2.3 Information Processing

Common to the difference-of-opinion literature, agents do not learn from prices nor do they learn from each other. They are dogmatic in their beliefs but behave rationally within them. *They are who they are* and no "rational" agent in the rational expectations sense exists.

Agents update their beliefs about $\mu_t^i,\ i \neq j;\ i,j \in \{A,B\}$ using the continuous time version of Bayes' rule using the combination of realized outputs for asset i,j and $\mathbf{d}s_t^i$ where appropriate[3]. Applying Theorem 12.7 from Liptser and Shiryaev (2000), the generalist up-

[3] In the interest of exposition, I exclude the behavior of specialist 2 (who covers asset B) and sometimes asset B, given the symmetry to specialist 1.

dates his beliefs according to[4]

$$\mathbf{d}\hat{\mu}_A^G = -\zeta(\hat{\mu}_A^G - \bar{\mu})\mathbf{d}t + \overbrace{\frac{\Omega^G - \rho\tilde{\Omega}^G}{\sigma^2(1-\rho^2)}}^{(\approx.08)} \underbrace{\left(\frac{\mathbf{d}X^A}{X^A} - \hat{\mu}_A^G\mathbf{d}t\right)}_{\substack{\text{adjustment for}\\ \text{forecast error}\\ \text{about } A}} + \underbrace{\frac{\tilde{\Omega}^G - \rho\Omega^G}{\sigma^2(1-\rho^2)}}_{(\approx-.035)} \overbrace{\left(\frac{\mathbf{d}X^B}{X^B} - \hat{\mu}_B^G\mathbf{d}t\right)}^{\substack{\text{adjustment for}\\ \text{forecast error}\\ \text{about } B}} + \phi'\sigma_\mu\mathbf{d}s^A \tag{3}$$

where Ω^G is the posterior variance of the forecast and $\tilde{\Omega}^G$ is the covariance of forecasts between $A\&B$. Lastly, $\phi'\sigma_\mu\mathbf{d}s^A$ is the adjustment using the signal about A and we can see how ϕ' modifies the sensitivity to the signal. I include as a reference the slope of adjustment when $\rho = 0.5,\ \sigma = 0.15,\ \sigma_\mu = 0.02,\ \zeta = 0.2,\ \phi' = 0.4,\ \phi = 0.6$ and omit the generalist's update for B given the symmetry.

Because of output correlation, we can see from (3) that the generalist updates his belief about $\mathbf{d}\hat{\mu}_A^G$ using the output of asset B. The logic is this: when there is a positive surprise in the output of $\mathbf{d}X_t^B$, the generalist *lowers* his forecast of μ_t^A to account for the possibility that a large $\mathbf{d}Z_t^B$ transmits to the output of $\mathbf{d}X_t^A$ in a small way[5], and we can compare the slopes of adjustment to reinforce this point.

In contrast, specialist 1 updates his beliefs according to

$$\mathbf{d}\hat{\mu}_A^1 = -\zeta(\hat{\mu}_A^1 - \bar{\mu})\mathbf{d}t + \overbrace{\frac{\Omega_A^1 - \rho\tilde{\Omega}^1}{\sigma^2(1-\rho^2)}}^{(\approx.057)} \left(\frac{\mathbf{d}X^A}{X^A} - \hat{\mu}_A^1\mathbf{d}t\right) + \underbrace{\frac{\tilde{\Omega}^1 - \rho\Omega_A^1}{\sigma^2(1-\rho^2)}}_{(\approx-.024)} \left(\frac{\mathbf{d}X^B}{X^B} - \hat{\mu}_B^1\mathbf{d}t\right) + \phi\sigma_\mu\mathbf{d}s^A$$

$$\mathbf{d}\hat{\mu}_B^1 = -\zeta(\hat{\mu}_B^1 - \bar{\mu})\mathbf{d}t + \overbrace{\frac{\Omega_B^1 - \rho\tilde{\Omega}^1}{\sigma^2(1-\rho^2)}}^{(\approx.104)} \left(\frac{\mathbf{d}X^B}{X^B} - \hat{\mu}_B^1\mathbf{d}t\right) + \underbrace{\frac{\tilde{\Omega}^1 - \rho\Omega_B^1}{\sigma^2(1-\rho^2)}}_{(\approx-.047)} \left(\frac{\mathbf{d}X^A}{X^A} - \hat{\mu}_A^1\mathbf{d}t\right) \tag{4}$$

Readily, the inequality $\phi > \phi'$ leads to the intuitive answer that relative to the generalist, specialist 1 under-reacts to surprises in output when updating $\mathbf{d}\hat{\mu}_A$ because he assigns a

[4]I suppress time subscripts here and begin to abuse notation, but it should be clear that $\hat{\mu}_t$ changes over time.

[5]Conversely, how a very large $\mathbf{d}Z_t^A$ transmits to $\mathbf{d}X_t^B$ in a small way.

higher weight to the informativeness of $\mathbf{d}s_t^A$. The trade-off is that he overreacts to surprises in output when updating $\mathbf{d}\hat{\mu}_B$ given the greater posterior variance of his forecast $\Omega_B^1 > \Omega_A^1$.

Van Nieuwerburgh and Veldkamp (2009) address the paradox that in order for an investor to diversify his portfolio, he must learn about the payoffs of risky securities. But the more he learns about a specific security, the greater his precision and informational advantage become and as a result, the investor trades more of it − leading to under-diversification. Similarly in my model, specialist 1 trades asset A more aggressively than both the generalist and specialist 2 do. Conversely, specialist 2 trades asset B most aggressively. The generalist, though better informed about both assets, is sandwiched between two aggressive specialists. This is the primary tension that at low levels of precision prevents knowing many noisy signals to be conducive towards survival in financial markets.

To visually illustrate how (3) and (4) differ, I first present a useful result.

Lemma 1. The dynamics of the forecast error between the generalist's estimates and the true conditional mean are governed by

$$\mathbf{d}f_i^G = -\left(\zeta + \frac{\Omega^G - \rho\tilde{\Omega}^G}{\sigma^2\left(1-\rho^2\right)}\right) f_i^G \mathbf{d}t - \frac{\left(\tilde{\Omega}^G - \rho\Omega^G\right)}{\sigma^2\left(1-\rho^2\right)} f_j^G \mathbf{d}t + \left(\frac{\left(\Omega^G - \rho\tilde{\Omega}^G\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_p + \frac{\left(\tilde{\Omega}^G - \rho\Omega^G\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_m\right)\mathbf{d}Z_t^i$$
$$+ \left(\frac{\left(\Omega^G - \rho\tilde{\Omega}^G\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_m + \frac{\left(\tilde{\Omega}^G - \rho\Omega^G\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_p\right)\mathbf{d}Z_t^j + \phi'\sigma_\mu \mathbf{d}s_t^i - \sigma_\mu \mathbf{d}B_t^i$$

for assets $i \neq j;\ i, j \in \{A, B\}$ and where

$$\sigma_p = \sigma\sqrt{\frac{1+\sqrt{1-\rho^2}}{2}}; \quad \sigma_m = \sigma\sqrt{\frac{1-\sqrt{1-\rho^2}}{2}}$$

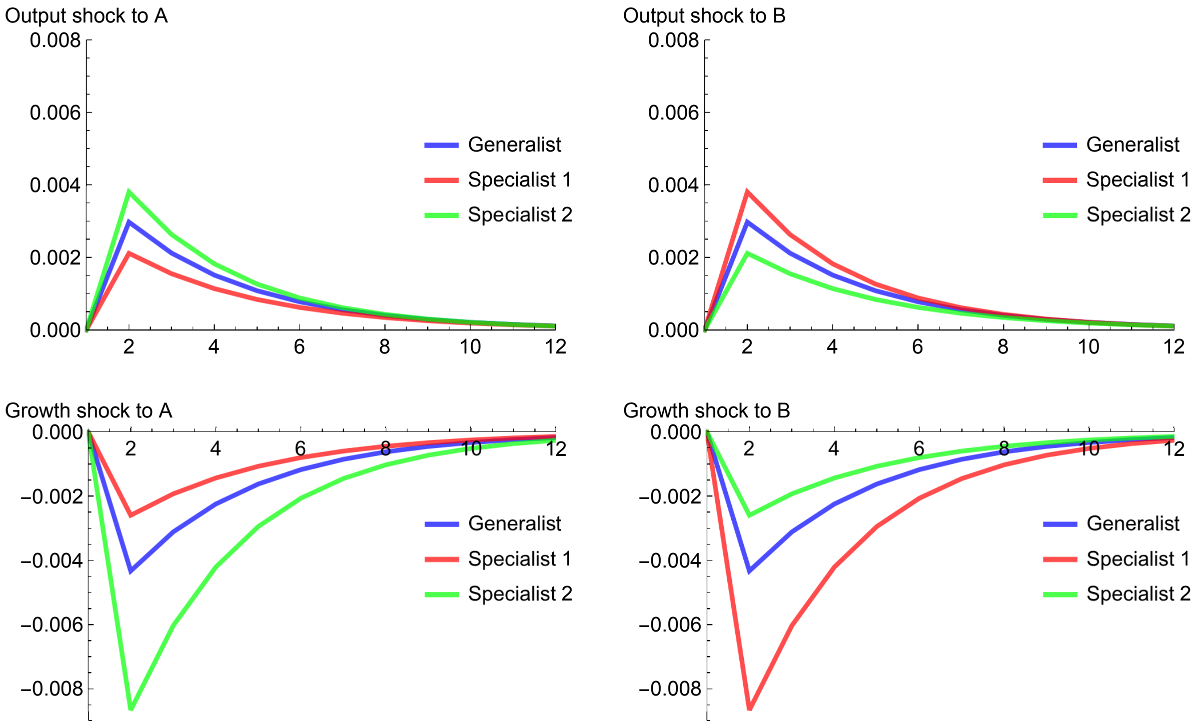


Figure 1: **Impulse response of an agent's forecast error in response to a unit Brownian shock at $t = 1$ for $\phi' = 0.5$, $\phi = 0.7$.** Forecasts about μ^A (left), μ^B (right). For reference, 0.004 on the y-axis corresponds to a forecast error of 40 basis points at the monthly horizon. With $\sigma_\mu = 0.03/\sqrt{12}$, the size of a growth shock is about 87 basis points.

Likewise for specialist 1,

$$\begin{aligned}
\mathbf{d}f_A^1 = &-\left(\zeta + \frac{\Omega_A^1 - \rho\tilde{\Omega}^1}{\sigma^2\left(1-\rho^2\right)}\right) f_A^1 \mathbf{d}t - \frac{\left(\tilde{\Omega}^1 - \rho\Omega_A^1\right)}{\sigma^2\left(1-\rho^2\right)} f_B^1 \mathbf{d}t + \left(\frac{\left(\Omega_A^1 - \rho\tilde{\Omega}^1\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_p + \frac{\left(\tilde{\Omega}^1 - \rho\Omega_A^1\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_m\right)\mathbf{d}Z_t^A \\
&+ \left(\frac{\left(\Omega_A^1 - \rho\tilde{\Omega}^1\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_m + \frac{\left(\tilde{\Omega}^1 - \rho\Omega_A^1\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_p\right)\mathbf{d}Z_t^B + \phi\sigma_\mu \mathbf{d}s_t^A - \sigma_\mu \mathbf{d}B_t^A \\
\mathbf{d}f_B^1 = &-\left(\zeta + \frac{\Omega_B^1 - \rho\tilde{\Omega}^1}{\sigma^2\left(1-\rho^2\right)}\right) f_B^1 \mathbf{d}t - \frac{\left(\tilde{\Omega}^1 - \rho\Omega_B^1\right)}{\sigma^2\left(1-\rho^2\right)} f_A^1 \mathbf{d}t + \left(\frac{\left(\Omega_B^1 - \rho\tilde{\Omega}^1\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_m + \frac{\left(\tilde{\Omega}^1 - \rho\Omega_B^1\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_p\right)\mathbf{d}Z_t^A \\
&+ \left(\frac{\left(\Omega_B^1 - \rho\tilde{\Omega}^1\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_p + \frac{\left(\tilde{\Omega}^1 - \rho\Omega_B^1\right)}{\sigma^2\left(1-\rho^2\right)}\sigma_m\right)\mathbf{d}Z_t^B - \sigma_\mu \mathbf{d}B_t^B
\end{aligned}$$

Proof. Define $f_i^G \doteq \hat{\mu}_i^G - \mu_i$ (at time t). Apply Itô's lemma to both sides, substituting in (3) and (2) into $\mathbf{d}\hat{\mu}_t^i$ and $\mathbf{d}\mu_t^i$ respectively. Same approach for specialist 1. □

Figure 1 shows how each agent's forecast deviates from the true conditional mean in response to a unit shock to one of $\{\mathbf{d}Z_t^A, \mathbf{d}Z_t^B, \mathbf{d}B_t^A, \mathbf{d}B_t^B\}$ when $\phi' = 0.5,\ \phi = 0.7$. The left panel shows the forecast error for μ^A and the right panel shows the forecast error for μ^B[6]. With generality, agents overestimate the true conditional mean in response to an output shock and underestimate in response to a growth shock. The overestimation in response to an output shock is muted; the minor differences in height is due to the different posterior variance each agent has over a forecast about asset A or B.

Clearly, the real story occurs in response to growth shocks. Focusing on the lower left panel, we see that specialist 2 is entirely blindsided by the growth shock to A because he ignores $\mathbf{d}s_t^A$. The fast decay afterwards is due to the inherent mean reversion and importantly, the adjustment in response to the lack of "noise" in $\mathbf{d}X_t^B/X_t^B$. In contrast, specialist 1 underestimates the true mean by 30% for $\phi = 0.7$. Still, he is the most *optimistic* about asset A relative to the others; specialist 1 places higher probabilities to good states of nature and we will see shortly in equilibrium how these differences in forecasts manifest in consumption paths.

Finally, another useful result:

Lemma 2. The change of measure, or the Radon-Nikodým derivative, that translates subjective probabilities between the generalist and specialist 1 is given by

$$\frac{\mathbf{d}\eta^1}{\eta^1} = -\begin{bmatrix} g_A^1 & g_B^1 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1+\sqrt{1-\rho^2}}{2}} & \sqrt{\frac{1-\sqrt{1-\rho^2}}{2}} \\ \sqrt{\frac{1-\sqrt{1-\rho^2}}{2}} & \sqrt{\frac{1+\sqrt{1-\rho^2}}{2}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}Z_A^G \\ \mathbf{d}Z_B^G \end{bmatrix} \tag{5}$$

where $g_A^1 \doteq \hat{\mu}_A^G - \hat{\mu}_A^1$ and $g_B^1 \doteq \hat{\mu}_B^G - \hat{\mu}_B^1$ and $\mathbf{d}Z_A^G, \mathbf{d}Z_B^G$ are the subjective Brownian motions of the generalist.

Proof. Since the two agents observe the same output, it is true that

$$\hat{\mu}_A^G \mathbf{d}t + \sigma_p \mathbf{d}Z_A^G + \sigma_m \mathbf{d}Z_B^G = \hat{\mu}_A^1 \mathbf{d}t + \sigma_p \mathbf{d}Z_A^1 + \sigma_m \mathbf{d}Z_B^1$$
$$\hat{\mu}_B^G \mathbf{d}t + \sigma_m \mathbf{d}Z_A^G + \sigma_p \mathbf{d}Z_B^G = \hat{\mu}_B^1 \mathbf{d}t + \sigma_m \mathbf{d}Z_A^1 + \sigma_p \mathbf{d}Z_B^1$$

[6] For exposition, I omit 4 plots that would make the figure complete, i.e. the forecast of μ^B in response to a $\mathbf{d}Z_t^A$ shock.

Collecting in matrix form,

$$\begin{bmatrix} \sigma_p & \sigma_m \\ \sigma_m & \sigma_p \end{bmatrix} \begin{bmatrix} \mathbf{d}Z_A^G \\ \mathbf{d}Z_B^G \end{bmatrix} = \begin{bmatrix} -g_A^1 \\ -g_B^1 \end{bmatrix} \mathbf{d}t + \begin{bmatrix} \sigma_p & \sigma_m \\ \sigma_m & \sigma_p \end{bmatrix} \begin{bmatrix} \mathbf{d}Z_A^1 \\ \mathbf{d}Z_B^1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{d}Z_A^G \\ \mathbf{d}Z_B^G \end{bmatrix} = - \begin{bmatrix} \sigma_p & \sigma_m \\ \sigma_m & \sigma_p \end{bmatrix}^{-1} \begin{bmatrix} g_A^1 \\ g_B^1 \end{bmatrix} \mathbf{d}t + \begin{bmatrix} \mathbf{d}Z_A^1 \\ \mathbf{d}Z_B^1 \end{bmatrix}$$

Now, define η_t^1 as a strictly positive random variable that translates beliefs specialist 1's has about the probability of events to the generalist's beliefs about the same events,

$$\mathbb{E}_t^1 \left[\mathbf{1}_{\{e_u\}} \right] = \mathbb{E}_t^G \left[\frac{\eta_u^1}{\eta_t^1} \mathbf{1}_{\{e_u\}} \right]$$

in which e_u is an event at time u. Applying Girsanov's theorem, we arrive at (5). $\square$

2.4 Equilibrium

Despite only having two risky assets, each in unit supply, and a riskless bond in zero net supply, I assume complete markets by synthesizing arbitrary futures contracts to complete the market. Individuals maximize lifetime utility and the generalist's problem is given by

$$\begin{aligned} &\sup_c \mathbb{E}^G \int_0^\infty e^{-\varrho t} \frac{(c_t^G)^{1-\gamma}}{1-\gamma} \mathbf{d}t \\ \text{st.} \qquad &\mathbb{E}^G \int_0^\infty \xi_t c_t^G \mathbf{d}t = \bar{\theta}^G \mathbb{E}^G \int_0^\infty \xi_t X_t \mathbf{d}t \end{aligned}$$

in which γ is the coefficient of relative risk aversion, ϱ is the time rate of preference, ξ_t is the stochastic discount factor in the economy, $\bar{\theta}^G$ is the generalist's endowed claim to endowment streams and $X_t = X_t^A + X_t^B$. Notice that expectations are set with respect to the generalist's subjective probability beliefs $\mathbb{E}^G$, and I use $\mathbb{E}^G$ as the reference measure throughout.

Specialist 1 faces an identical problem, except now we must apply the change of measure since prices must be set from a common reference point. Thus, specialist 1's individual maximization is

$$\sup_{c} \mathbb{E}^G \int_0^\infty \eta_t^1 e^{-\varrho t} \frac{(c_t^1)^{1-\gamma}}{1-\gamma} \mathbf{d}t$$
$$\text{st.} \qquad \mathbb{E}^G \int_0^\infty \xi_t c_t^1 \mathbf{d}t = \bar{\theta}^1 \mathbb{E}^G \int_0^\infty \xi_t X_t \mathbf{d}t$$

emphasizing η_t^1 now in the objective function. Taking first order conditions in each agent's maximization problem,

$$c_t^{G*} = \left(\lambda^G \xi_t e^{\varrho t}\right)^{-\frac{1}{\gamma}}; \quad c_t^{1*} = \left(\frac{\lambda^1 \xi_t e^{\varrho t}}{\eta^1}\right)^{-\frac{1}{\gamma}}; \quad c_t^{2*} = \left(\frac{\lambda^2 \xi_t e^{\varrho t}}{\eta^2}\right)^{-\frac{1}{\gamma}}$$

where λ is the Lagrange multiplier on the budget constraint, and imposing market clearing conditions in the goods market,

$$\left(\lambda^G \xi_t e^{\varrho t}\right)^{-\frac{1}{\gamma}} + \left(\frac{\lambda^1 \xi_t e^{\varrho t}}{\eta^1}\right)^{-\frac{1}{\gamma}} + \left(\frac{\lambda^2 \xi_t e^{\varrho t}}{\eta^2}\right)^{-\frac{1}{\gamma}} = X_t^A + X_t^B = X_t$$

the stochastic discount factor in this economy is given by

$$\xi_t = e^{-\varrho t} \left[\left(\frac{1}{\lambda^G}\right)^{\frac{1}{\gamma}} + \left(\frac{\eta_t^1}{\lambda^1}\right)^{\frac{1}{\gamma}} + \left(\frac{\eta_t^2}{\lambda^2}\right)^{\frac{1}{\gamma}} \right]^{\gamma} X_t^{-\gamma} \tag{6}$$

and the generalist's consumption c_t^{G*} is

$$c_t^{G*} = \underbrace{\frac{\left(\frac{1}{\lambda^F}\right)^{\frac{1}{\gamma}}}{\left(\frac{1}{\lambda^F}\right)^{\frac{1}{\gamma}} + \left(\frac{\eta_t^1}{\lambda^1}\right)^{\frac{1}{\gamma}} + \left(\frac{\eta_t^2}{\lambda^2}\right)^{\frac{1}{\gamma}}}}_{\doteq \omega_t(\eta_t^1, \eta_t^2)} X_t$$

and the important variable $\omega_t(\eta_t^1, \eta_t^2)$ is the generalist's share of consumption from the endowment at time t.

I now explain how changes in consumption occurs. Recall equation (5) and I re-state below for convenience.

$$\frac{\mathbf{d}\eta^1}{\eta^1} = -\begin{bmatrix} g_A^1 & g_B^1 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1+\sqrt{1-\rho^2}}{2}} & \sqrt{\frac{1-\sqrt{1-\rho^2}}{2}} \\ \sqrt{\frac{1-\sqrt{1-\rho^2}}{2}} & \sqrt{\frac{1+\sqrt{1-\rho^2}}{2}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}Z_A^G \\ \mathbf{d}Z_B^G \end{bmatrix}$$

Suppose there is an output shock to asset A. The generalist updates his belief about μ_A slightly more than specialist 1 does due to the specialist's superior precision. As a result, the generalist assigns higher probability to the likelihood that the next output will be higher and he bets on it through η^1 as $g_A^1 > 0$. If beliefs are confirmed by the next output shock, $\mathbf{d}\eta_t^1 < 0 \Rightarrow \eta^1 \downarrow$ and consequently, the generalist's share of consumption $\omega_t(\eta_t^1, \eta_t^2)$ increases. In other words, an agent can have incorrect beliefs but still be rewarded, and an agent can have the correct beliefs but luck works against him. Thus, the purpose of updating the expected returns of each asset is to make "informed" bets, but chance governs how the claim to endowment changes.

3 Results

Following convention in the literature on survival, I start with a basic definition.

Definition 1. An agent *vanishes*, or is *driven out of the market* on path s, if his share of consumption converges to zero. An agent *survives* if he is not driven out.

It is important to point out that it is the share of consumption, not wealth, that must converge to zero because it implies that the agent consistently made the worst forecasts. Moreover, since agents have time additive utility and I assumed complete markets, the results from Blume and Easley (2006) apply and the definition above is valid for this economy. I now state my main result.

Result 1. Survival. Suppose $\phi' \ll \phi$ and $\rho \neq 0, \gamma > 1$. Specialists are expected to *dominate* in the long-run if

I their lead over the generalist's skill is sufficiently large

II the generalist's skill is sufficiently low.

Otherwise, $\mathbb{E}[1-\omega_t]$ gets arbitrarily close to zero as $\phi' \to 1$.

Proof. See Appendix. □

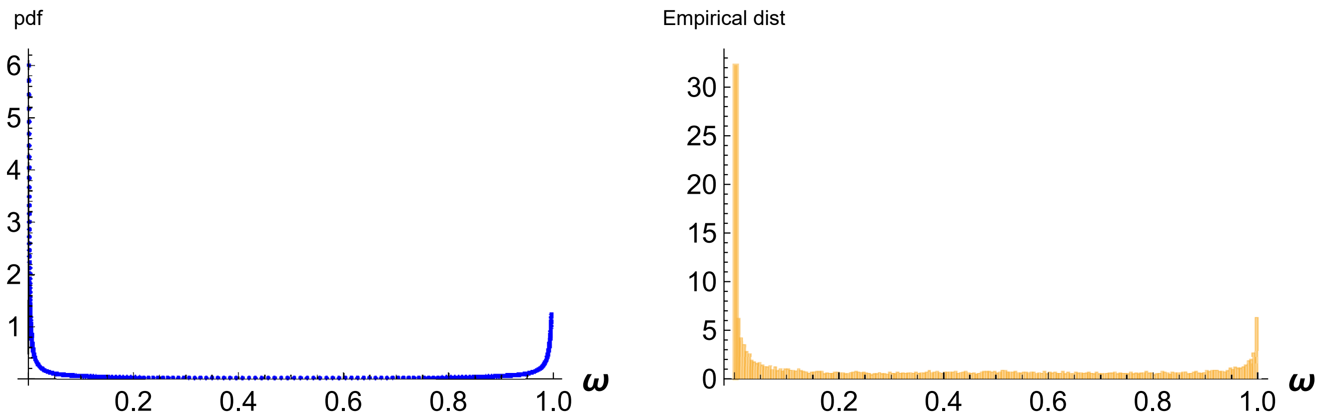


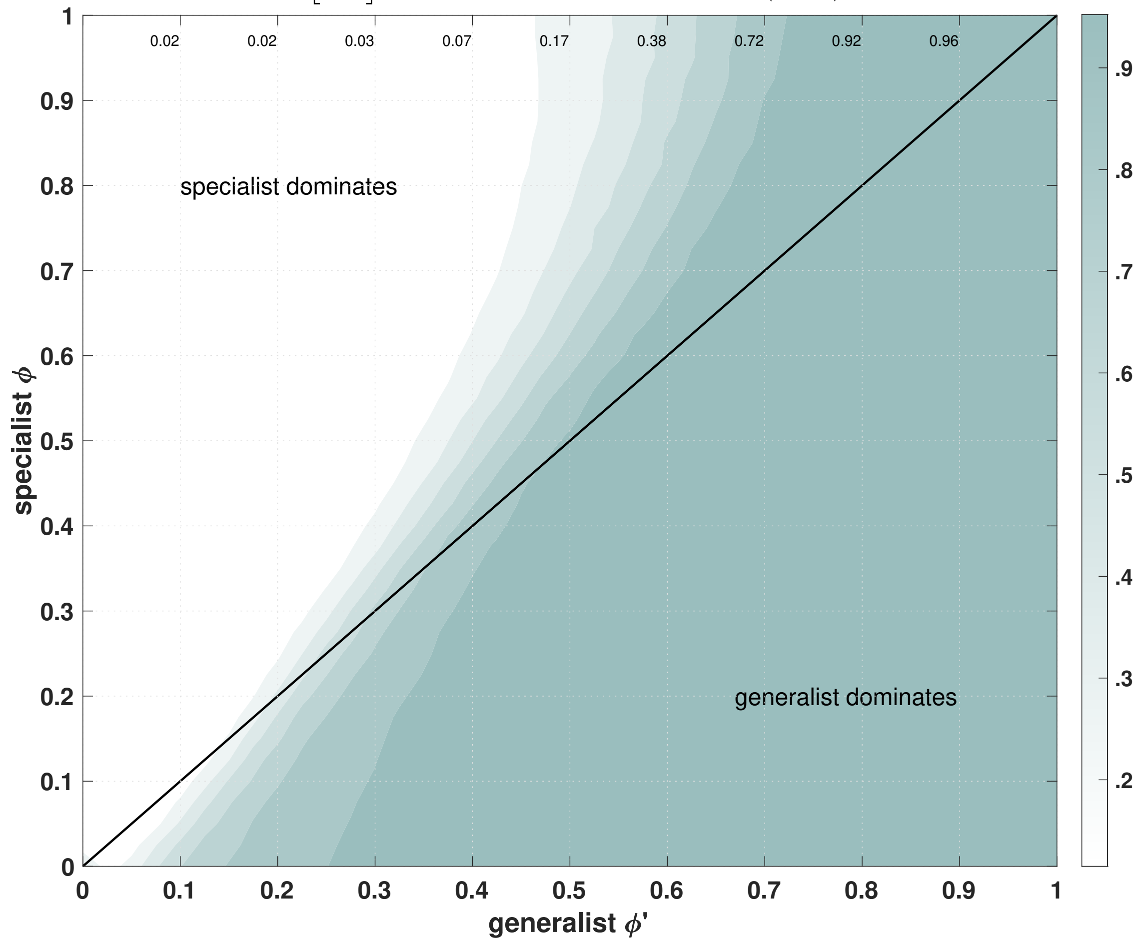
Figure 2: **50 year probability density function of the generalist's share of consumption (left) and empirical distribution of 20,000 sample paths (right) for $\phi' = 0.4$ and $\phi = 0.6$.** The pdf is calculated by first finding the characteristic function of $\mathbb{E}^P[(\eta_T^1)^\chi(\eta_T^2)^\epsilon]$ for $\chi, \epsilon \in \mathbb{C}$ and applying Fourier inversion to compute $\mathbb{E}^P[\omega_T]$. Simulations are conducted at monthly intervals for parameters when annualized, $\sigma = 0.15, \sigma_\mu = 0.03, \bar{\mu} = 0.015$ with risk-aversion $\gamma = 3$, output correlation $\rho = 0.5$ and speed of mean reversion $\zeta = 0.2$.

The left plot in figure (2) shows the probability density function of the generalist's share of consumption in 50 years when $\phi' = 0.4$ and $\phi = 0.6$. What this plot shows is that probabilistically, the generalist is expected to disproportionately vanish evident by the concentration at zero. About 17% of the time, the generalist is expected to luck out against both specialists and have a consumption share of 1. The right plot shows simulated sample paths to check that the empirical distribution matches the density function.

Figure (3) shows the same plot but now spanned over different combinations of precision. The generalist's skill is on the x-axis and the specialist's skill is on the y-axis. The colormap and the number on the ticks at the top axis, give the range of values for $\mathbb{E}^P[\omega_T(\eta_T^1, \eta_T^2)]$ in that band. Looking at the point $(0.4, 0.6)$, we confirm the color region expresses what is shown in Figure (2)[7]. Further, we look above the diagonal given the constraint $\phi > \phi'$.

[7]To be precise, the expectation is taken over the density "on the edges" at .05 cutoffs to avoid the pull

Figure 3: **Dominance Regions and $\mathbb{E}^P[\omega_T]$ on the Edges.** The color corresponds to the range of values for $\mathbb{E}^P[\omega_T]$ for a given coordinate pair (ϕ', ϕ).



An intuitive answer that is confirmed by the plot is that a very skilled generalist is optimal; Admati and Pfleiderer (1987) argue the same point by showing that signals are complements at high levels of precision in the setup I described in the introduction. What is not obvious however is that at low levels of precision, even when we are on the diagonal so that $\phi' = \phi$, the generalist is wiped out. In other words, the generalist has twice the information as a specialist but vanishes from the market. When signals are noisy, more information *harms* the investor, in the perspective of survival. This result is in contrast to the Monotone Likelihood Ratio Property in information economics in which an agent can't possibly be worse off with more information. In my economy however, more information

towards the middle. The expectation of $\omega_T(\eta_T^1, \eta_T^2)]$ in Figure (2) is actually .34, but the reader may agree that that number does not accurately describe the spirit of the plot.

only sandwiches the agent in-between two aggressive traders.

The intuition for why this happens has been alluded to in Figure (1). To see it more clearly, Figures (B.1) and (B.2) in the Appendix show the same impulse response functions, but for ($\phi' = 0.2, \phi' = 0.3$) and ($\phi' = 0.8, \phi = 0.9$). The reason is this: suppose there is a positive growth shock to asset A and suppose that in the next period, the output is indeed larger so that the bet turned out to be a good bet. For the generalist to survive, his cumulative net gains (what he loses to specialist 1 plus his gains over specialist 2) must be large enough to be able to absorb net losses if the bet turns out wrong − for example if $\mathbf{d}Z_t^A$ is large and negative and offsets the positive $\mathbf{d}B_t^A$ that agents observed through $\mathbf{d}s_t^A$. The friction is that the generalist must split his gains but incurs the full loss given the asymmetry of trades as Figure (1) shows[8]. To resolve this problem, the generalist wants to be as close to specialist 1 as possible while being as far away from specialist 2 as possible in beliefs about A.

The role of ϕ' then is to create the separation that the generalist needs to have meaningful gains, which can be seen comparing (B.1) to (B.2). Because the information asymmetry is increasing in ϕ' (since a specialist ignores one signal), the magnitude of gains and losses become larger and the ratio of losses/gains gets smaller. At low levels of precision, there are almost no differences in beliefs but the generalist splits small gains and lose an amount that is proportionally large (for example, the ratio of losses/gains to the share of consumption is about 2 at low precision but approaches 1 at higher levels.). Being informed about the other asset only accelerates this process. It is not until the ratio of losses/gains becomes sufficiently small that probabilistically, knowing about two assets works to his favor.

3.1 Comparative Statics

The same logic could be applied to understand the effects of risk-aversion.

Result 2. Risk aversion. Fix $\phi' \ll \phi$ and $\rho \neq 0$. Then, there is a $\boldsymbol{\gamma}^*$ such that $\lim_t \mathbb{E}[\boldsymbol{\omega}_t] = 0$ for any ϕ'.

Output Correlation. Suppose $\phi' \ll \phi$ and $\boldsymbol{\gamma} > 1$. As $\rho \to 1$ the dominance region shifts

[8] Since this is a 3-person economy, there will always be an uneven number of buyers/sellers

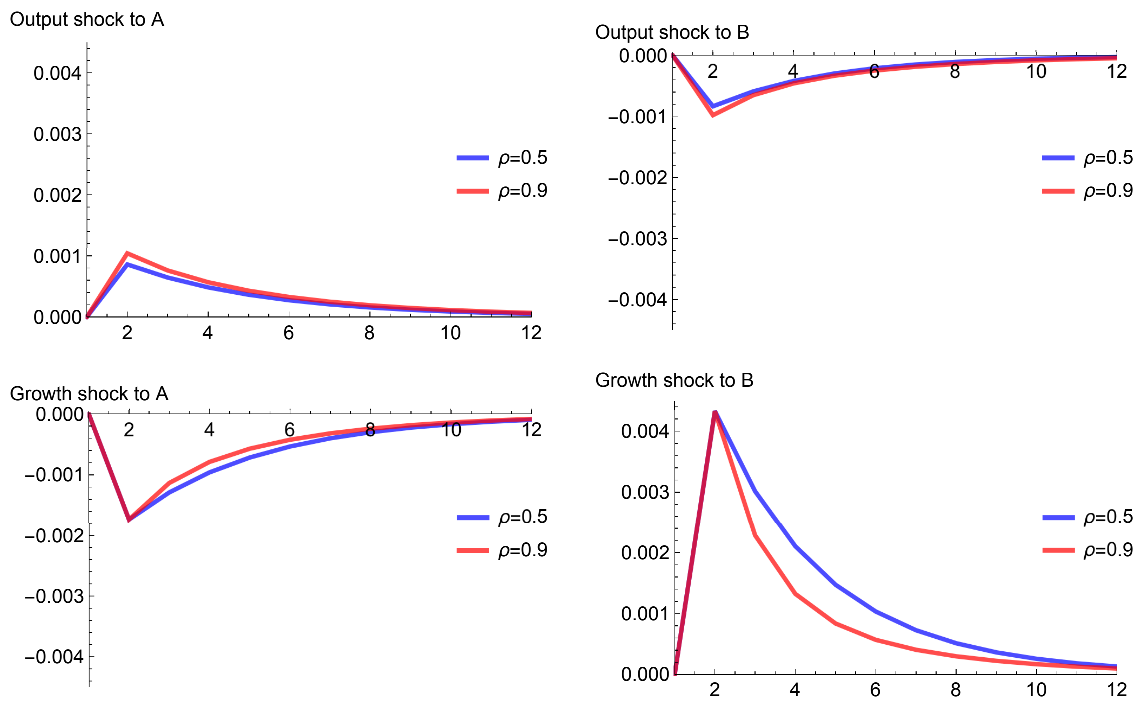


Figure 4: **Impulse response of the disagreement between forecasts in response to an output shock between the generalist and specialist 1.** The plots show $g_A^1 \doteq \hat{\mu}_A^F - \hat{\mu}_A^1$ (left panel) and $g_B^1 \doteq \hat{\mu}_B^F - \hat{\mu}_B^1$ (right panel).

to the right, and vice versa

I do not include a proof but setting $\gamma = 200$ effectively shows the first result (the plot in Figure (3) would just be white), though it is expected to take *200* years for the generalist to vanish, not 50. The intuition is that there are only small pockets of opportunity for the generalist to make money. Risk-aversion restrains the generalist from exploiting a specialist fully when there are growth shocks to the other asset. If the generalist is too risk-averse, then the gains are never decisive for the generalist to get to the finish line and though it would take 200 years, the generalist will just eventually toil away from everyday Brownian shocks.

Output correlation is also straightforward. If there is no correlation, the specialist is at a disadvantage since he has no information about the other asset. Conversely, if correlation goes to 1, even though specialist 1 does not see the growth shock to B, the high signal correlation accelerates his recovery, as the inward push of the curvature in Figure (4) lower right panel shows. In other words, the specialist is able to adjust faster and the generalist loses the space that affords him to take advantage.

3.2 Business Cycles

In the final subsection, I study a recurring theme that generalists are able to time the market. The first point of clarification is as already mentioned, the definition of generalist appears to differ between knowing many things versus knowing about the market − though I would argue that a specialist could specialize in the market. This paper can only speak to a generalist as defined as knowing many things.

In this paper, generalists also thrive in recessions *and* expansion − a symmetry empirically discussed in Kacperczyk et al. (2014). I start with a definition and present my final result, which admittedly is more of an informed hypothesis.

Definition 2. The economy is in an expansion (recession) if the conditional rate of growth μ_t^i is more than 1 standard deviation above (below) the average $\bar{\mu}$.

Result 3. Business Cycles. Suppose a path s is dominated by extreme events. Moreover, assume the generalist is sufficiently skilled, $\phi' > \phi^*$. Then, more often than not, specialists are driven out of the market even for low $\mathbb{E}[\omega_t]$.

I now present another useful result.

Lemma 3. The Malliavin derivative[9], or the impact of an instantaneous shock today on the

[9] To help motivate the use of Malliavin calculus, consider the solution to the stochastic differential equation of the equivalent martingale measure $\mathrm{d}\xi_t = -\xi_t r_t \mathbf{d}t - \xi_t \kappa_t \mathbf{d}W_t$ where $\kappa = (\mu_t - r_t)/\sigma_t$,

$$\xi_t = \xi_0 \exp\left(-\int_0^t (r_s + \frac{1}{2}\kappa_s^2)\mathrm{d}s - \int_0^t \kappa_s \mathrm{d}W_s\right)$$

The Malliavin derivative, which is a partial derivative, is the impact of an innovation in the Brownian motion $\mathbf{d}W_t$ at time t on the stochastic discount factor at time T. Applying the chain rule of Malliavin calculus,

$$\frac{\mathscr{D}_t \xi_T}{\xi_T} = -\kappa_t$$

which is why in Dumas et al. (2009),

> "the instantaneous market price of risk (or Sharpe ratio) is equal to minus the diffusion of the pricing measure. It is the instantaneous response of the stochastic discount factor to shocks occuring today."

See Appendix D in Detemple et al. (2003) for an excellent introduction.

share of consumption $\omega_T(\eta_T^1, \eta_T^2)$ at a later date T, is given by

$$\mathscr{D}_t \omega_T(\eta_T^1, \eta_T^2) = \frac{-\omega_t(\eta_t^1, \eta_t^2)\left[\frac{1}{\lambda^1\gamma}\left(\frac{\eta_T^1}{\lambda^1}\right)^{\frac{1}{\gamma}-1}\mathscr{D}_t\eta_T^1 + \frac{1}{\lambda^2\gamma}\left(\frac{\eta_T^2}{\lambda^2}\right)^{\frac{1}{\gamma}-1}\mathscr{D}_t\eta_T^2\right]}{\left[\left(\frac{1}{\lambda^G}\right)^{\frac{1}{\gamma}} + \left(\frac{\eta_T^1}{\lambda^1}\right)^{\frac{1}{\gamma}} + \left(\frac{\eta_T^2}{\lambda^2}\right)^{\frac{1}{\gamma}}\right]} \tag{7}$$

where

$$\begin{aligned}
\mathscr{D}_t\eta_T^1 = &\begin{bmatrix} -\frac{\sqrt{1+\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}}g_{A,t}^1 + \frac{\sqrt{1-\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}}g_{B,t}^1 \\ +\frac{\sqrt{1-\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}}g_{A,t}^1 - \frac{\sqrt{1+\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}}g_{B,t}^1 \end{bmatrix}^{\dagger} + \int_t^T \left(\frac{\rho}{\sigma^2(1-\rho^2)}g_{B,u}^1 - \frac{1}{\sigma^2(1-\rho^2)}g_{A,u}^1\right)\mathscr{D}_t g_{A,u}^1 \mathbf{d}u \\
&+ \int_t^T \left(\frac{\rho}{\sigma^2(1-\rho^2)}g_{A,u}^1 - \frac{1}{\sigma^2(1-\rho^2)}g_{B,u}^1\right)\mathscr{D}_t g_{B,u}^1 \mathbf{d}u \\
&+ \int_t^T \left(-\frac{\sqrt{1+\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}}\mathscr{D}_t g_{A,u}^1 + \frac{\sqrt{1-\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}}\mathscr{D}_t g_{B,u}^1\right)\mathbf{d}Z_u^A \\
&+ \int_t^T \left(\frac{\sqrt{1-\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}}\mathscr{D}_t g_{A,u}^1 - \frac{\sqrt{1+\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}}\mathscr{D}_t g_{B,u}^1\right)\mathbf{d}Z_u^B
\end{aligned}$$

and $\mathscr{D}_t g_{A,u}^1, \mathscr{D}_t g_{B,u}^1$ given in the appendix.

Proof. See Appendix. □

With these tools at hand, I now explain Figure (5) above. The top left panel shows the generalist's change in the share of consumption in response to a Z^A shock when asset A is either in a recession or an expansion and asset B is in normal times. The bottom left shows the same plot but for a Z^B shock. As Equation (7) shows, the effect of a shock is difficult to disentangle due to the various cross-interactions. However, the left panel essentially conveys that the losses are smaller than the gains. The condition $\phi' > \phi^*$ is to ensure that the gains on a Z^B shock is larger than the losses on a Z^A shock; when asset A is at a peak or a trough, the "turbulence" so to speak allows the generalist to exploit the information asymmetry in asset B much more than what specialist 1 can extract from the generalist.

The right panel further shows that when both assets are in expansions or recessions, the generalist always makes at least a small gain; the generalist's constant in-betweenness

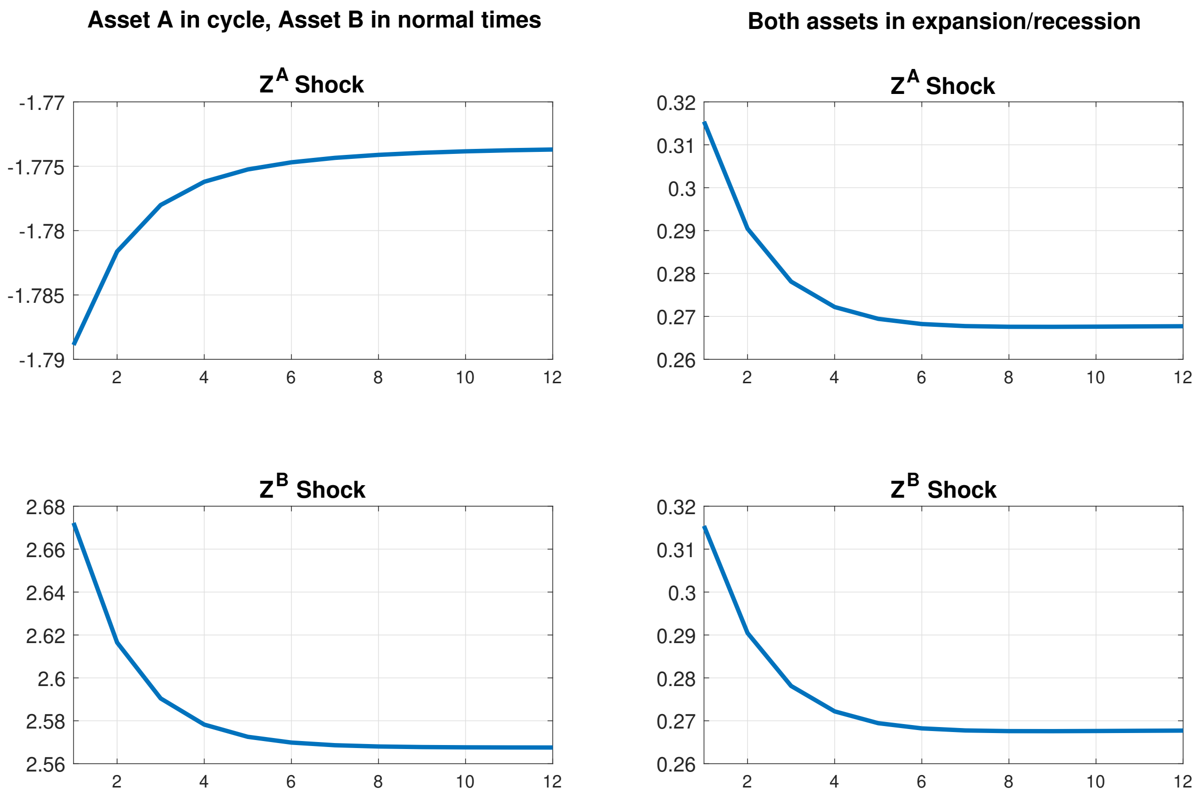


Figure 5: **Impulse response of the Δ in generalist's share of consumption in response to an output shock.** Plots on the LHS correspond to when only Asset A is in booms and busts. Y-axis in percentage points. $\phi' = 0.5, \phi = 0.7$

creates a buffer against both sides when disagreement is most volatile, avoiding large losses (comparatively) when the growth rate is away from its mean. Simulations confirm that when an asset is in an expansion/recession, the corresponding specialist's average mean-squared error of the *other* asset multiplies by a factor of 3. Finally, Figure (6) shows an empirical exercise, populating sample paths with Brownian motion that concentrate in business cycles. The figure shows that while the generalist's consumption share is still centered around 1/3 (his starting share) in the left panel, augmenting sample paths to be excessively volatile (in the disagreement sense) "clears" the market much sooner, and to the generalist's favor − recall that in Figure (2), the concentration was at zero.

From the 2 figures above, a plausible inference is that business cycles do not occur frequently enough for a non-high skilled generalist to be profitable.

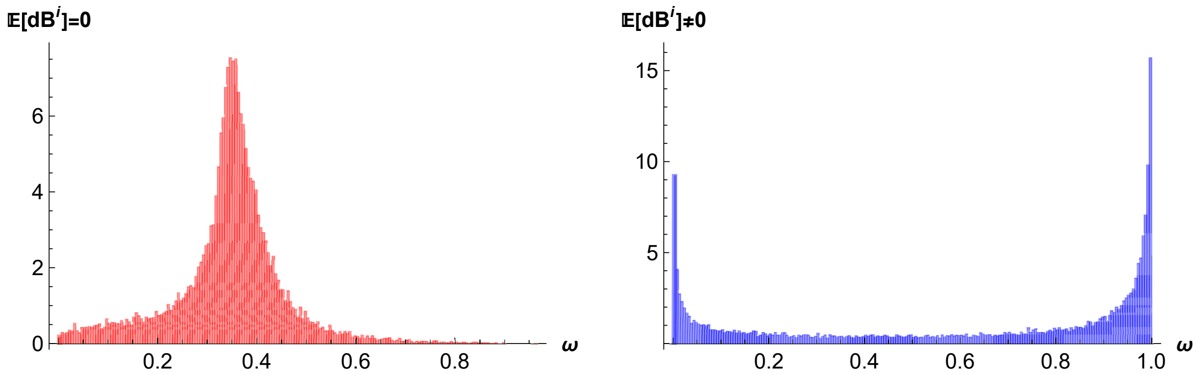


Figure 6: **15 year empirical distribution of the generalist's share of consumption when $\mathbb{E}[\mathbf{d}B_t^i] = 0$ (left) and $\mathbb{E}[\mathbf{d}B_t^i] \neq 0$ (right).** For illustration purposes, and parameters are the same as in Figure (2).

4 Conclusion

In a three-agent economy with two risky assets, the most reasonable conclusion is that an agent who is perpetually in-between two other agents in trade size is at a material disadvantage on average. In my paper, the generalist fulfilled that role as the trade-off for "knowing many things". Indeed, Figure (B.3) in the Appendix explores the case when the true informativeness of the signal is the midpoint of the precision between the generalist and the specialist − in other words, the specialist is "overconfident". If the signal is allowed to be truly noisy[10], the tentativeness of the generalist appears to be a bigger crutch; these figures raise the question on whether the market favors specialized traders. In the real world, it's more likely that we are in regime of many assets and many noisy signals. The results of my paper suggests that on average, specialists should perform better than generalists.

This paper differs from existing literature on information specialization by allowing generalists and specialists interact with each other. I argue that this distinction is important−the optimality of signals as complements or substitutes is of second order importance. How to position oneself against the other players in the market hold primacy.

[10]So the true process of a signal follows $\mathbf{d}s_t^A = \phi^* \mathbf{d}B_t^A + \sqrt{1-(\phi^*)^2}\mathbf{d}W_t^A$ in which ϕ^* is the midpoint.

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A Proofs

Proof of Result 1. The goal is to find $\mathbb{E}^P\left[\omega_T\right]$, the expected share of consumption at T under the true probability measure. To know this quantity, and given the dependence $\omega_t(\eta_t^1, \eta_t^2)$, we need to know the joint conditional distribution of η_u^1 and η_u^2 at a later date u under the physical measure, given the value of the state variables at time t. To formalize, we first look for the characteristic function, or the Fourier transform, of the unknown transition density

$$H\left(\eta^1, \eta^2, f_A^G, f_B^G, f_A^1, f_B^1, f_A^2, f_B^2, t, u; \chi, \epsilon\right) = \mathbb{E}_t^P\left[\left(\eta_u^1\right)^\chi \left(\eta_u^2\right)^\epsilon\right] \quad \chi, \epsilon \in \mathbb{C}$$

and once it is found, we can apply the inverse Fourier transform to get back the original probability density function. Now, the Feynman-Kac theorem tells us that the function $H\left(\eta^1, \eta^2, f_A^G, f_B^G, f_A^1, f_B^1, f_A^2, f_B^2, t, u; \chi, \epsilon\right)$ must satisfy the linear PDE:

$$0 = \mathcal{L}H\left(\eta^1, \eta^2, f_A^G, f_B^G, f_A^1, f_B^1, f_A^2, f_B^2, t, u; \chi, \epsilon\right) + \frac{\partial H}{\partial t}\left(\eta^1, \eta^2, f_A^G, f_B^G, f_A^1, f_B^1, f_A^2, f_B^2, t, u; \chi, \epsilon\right)$$

with initial condition $H\left(\eta^1, \eta^2, f_A^G, f_B^G, f_A^1, f_B^1, f_A^2, f_B^2, t, t; \chi, \epsilon\right) = (\eta^1)^\chi(\eta^2)^\epsilon$ and where $\mathcal{L}$ is the differential generator of the state variables. The function arguments for H is due to the fact that we must find the density with respect to the physical measure. Furthermore, η^1 is now given by

$$\begin{aligned}\frac{\mathbf{d}\eta^1}{\eta^1} = &\left[(f_A^G - f_A^1) \quad (f_B^G - f_B^1)\right]\begin{bmatrix} \frac{1}{\sigma^2(1-\rho^2)} & -\frac{\rho}{\sigma^2(1-\rho^2)} \\ -\frac{\rho}{\sigma^2(1-\rho^2)} & \frac{1}{\sigma^2(1-\rho^2)} \end{bmatrix}\begin{bmatrix} f_A^G \\ f_B^g \end{bmatrix}\mathbf{d}t \\ &- \left[(f_A^G - f_A^1) \quad (f_B^G - f_B^1)\right]\begin{bmatrix} \frac{\sqrt{1+\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}} & -\frac{\sqrt{1-\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}} \\ -\frac{\sqrt{1-\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}} & \frac{\sqrt{1+\sqrt{1-\rho^2}}}{\sigma\sqrt{2(1-\rho^2)}} \end{bmatrix}\begin{bmatrix} \mathbf{d}Z^A \\ \mathbf{d}Z^B \end{bmatrix}\end{aligned}$$

which is derived by adding and subtracting the true conditional mean to the first step in the proof of Lemma 2 and making the substitution to change from the generalist's subjective Brownian motion to the true measure Brownians.

The differential generator $\mathcal{L}H\left(\eta^1, \eta^2, f_A^G, f_B^G, f_A^1, f_B^1, f_A^2, f_B^2, t, u; \chi, \epsilon\right)$ actually has over 25 terms, and the tedious algebra to get to the next step takes a dozen pages. For our collective sanity, I will write down the first few terms and explain what needs to happen.

$$
\begin{aligned}
&\mathcal{L}H\left(\eta^1,\eta^2,f_A^G,f_B^G,f_A^1,f_B^1,f_A^2,f_B^2,t,u;\chi,\epsilon\right) = \\
&\qquad \frac{\partial H}{\partial \eta^1}\left(\mathbf{d}\eta^1\right) + \frac{1}{2}\frac{\partial^2 H}{\partial (\eta^1)^2}\left(\mathbf{d}\eta^1\right)^2 + \cdots \text{ (for each state variable)} \\
&\qquad + \frac{\partial^2 H}{\partial \eta^1 \partial \eta^2}(\mathbf{d}\eta^1)(\mathbf{d}\eta^2) + \frac{\partial^2 H}{\partial \eta^1 \partial f_A^G}(\mathbf{d}\eta^1)(\mathbf{d}f_A^G) + \frac{\partial^2 H}{\partial \eta^1 \partial f_B^G}(\mathbf{d}\eta^1)(\mathbf{d}f_B^G) + \cdots \\
&\qquad + \frac{\partial^2 H}{\partial f_A^G \partial f_B^G}(\mathbf{d}f_A^G)(\mathbf{d}f_B^G) + \frac{\partial^2 H}{\partial f_A^G \partial f_A^1}(\mathbf{d}f_A^G)(\mathbf{d}f_A^1) + \frac{\partial^2 H}{\partial f_A^G \partial f_B^1}(\mathbf{d}f_A^G)(\mathbf{d}f_B^1) \\
&\qquad + \frac{\partial^2 H}{\partial f_A^G \partial f_A^2}(\mathbf{d}f_A^G)(\mathbf{d}f_A^2) + \cdots \text{ (all remaining combinations)}
\end{aligned}
$$

The solution to this PDE takes the exponential affine linear quadratic form,

$$
H\left(\eta^1,\eta^2,f_A^G,f_B^G,f_A^1,f_B^1,f_A^2,f_B^2,t,u;\chi,\epsilon\right) = (\eta^1)^\chi(\eta^2)^\epsilon H_P\left(f_A^G,f_B^G,f_A^1,f_B^1,f_A^2,f_B^2,t,u;\chi,\epsilon\right)
$$

where

$$
\begin{aligned}
H_P\left(f_A^G,f_B^G,f_A^1,f_B^1,f_A^2,f_B^2,t,u;\chi,\epsilon\right) = \exp\Big(&C(\chi,\epsilon;u-t) + C_A^G(\chi,\epsilon;u-t)(f_A^G)^2 \\
&+ C_B^G(\chi,\epsilon;u-t)(f_B^G)^2 + C_A^1(\chi,\epsilon;u-t)(f_A^1)^2 + C_B^1(\chi,\epsilon;u-t)(f_B^1)^2 \\
&+ C_A^2(\chi,\epsilon;u-t)(f_A^2)^2 + C_B^2(\chi,\epsilon;u-t)(f_B^2)^2\Big)
\end{aligned}
$$

Now to find the seven coefficients, one must solve the resulting system of 6 Riccati ordinary differential equations, and a seventh ODE of order 1 in which the coefficients are all functions of time; the reader may forgive that I omit writing down the reduced system. Thus, I explicitly derive the characteristic function up to the numerical solution of a system of Riccati equations, which can then be easily solved for in *Mathematica*.

Once the function H is found, one can apply the inverse Fourier transform to get back the original probability distribution function for the joint conditional distribution of η^1 and η^2. In particular,

$$
\mathbb{E}^P[\omega(\eta_u^1,\eta_u^2)] = \iint_0^\infty \omega(\eta^1,\eta^2)\left[\frac{1}{2\pi}\iint_{-\infty}^\infty \left(\frac{\tilde{\eta}^1}{\eta^1}\right)^{-i\chi}\left(\frac{\tilde{\eta}^2}{\eta^2}\right)^{-i\epsilon} H_P(\cdot)\, d\chi d\epsilon\right]\frac{d\tilde{\eta}^1}{\eta^1}\frac{d\tilde{\eta}^2}{\eta^2}
$$

One can then take the expected value over this distribution for a given pair of (ϕ', ϕ). $\square$

Proof of Lemma 3.

$$\mathscr{D}_t g^1_{A,T} = e^{-\psi^1_{(1,1)}(T-t)} \begin{bmatrix} \frac{(\Omega^G - \Omega^1_A) - \rho(\tilde{\Omega}^G - \tilde{\Omega}^1)}{\sigma^2(1-\rho^2)}\sigma_p + \frac{(\tilde{\Omega}^G - \tilde{\Omega}^1 - \rho(\Omega^G - \Omega^1_A))}{\sigma^2(1-\rho^2)}\sigma_m \\ \frac{(\Omega^G - \Omega^1_A) - \rho(\tilde{\Omega}^G - \tilde{\Omega}^1)}{\sigma^2(1-\rho^2)}\sigma_m + \frac{(\tilde{\Omega}^G - \tilde{\Omega}^1 - \rho(\Omega^G - \Omega^1_A))}{\sigma^2(1-\rho^2)}\sigma_p \\ (\phi' - \phi)\sigma_\mu \\ \\ \end{bmatrix}^\dagger - \frac{\tilde{\Omega}^1 - \rho\Omega^1_A}{\sigma^2(1-\rho^2)} \int_t^{T-1} e^{-\psi^1_{(2,2)}(t-u)} \mathscr{D}_t g^1_{B,u} \mathbf{d}u$$

$$\mathscr{D}_t g^1_{B,T} = e^{-\psi^1_{(2,2)}(T-t)} \begin{bmatrix} \frac{(\Omega^G - \Omega^1_B) - \rho(\tilde{\Omega} - \tilde{\Omega}^1)}{\sigma^2(1-\rho^2)}\sigma_m + \frac{(\tilde{\Omega} - \tilde{\Omega}^1 - \rho(\Omega^G - \Omega^1_B))}{\sigma^2(1-\rho^2)}\sigma_p \\ \frac{(\Omega^G - \Omega^1_B) - \rho(\tilde{\Omega} - \tilde{\Omega}^1)}{\sigma^2(1-\rho^2)}\sigma_p + \frac{(\tilde{\Omega} - \tilde{\Omega}^1 - \rho(\Omega^G - \Omega^1_B))}{\sigma^2(1-\rho^2)}\sigma_m \\ \\ \phi'\phi_\mu \end{bmatrix}^\dagger - \frac{\tilde{\Omega}^1 - \rho\Omega^1_B}{\sigma^2(1-\rho^2)} \int_t^{T-1} e^{-\psi^1_{(1,1)}(t-u)} \mathscr{D}_t g^1_{A,u} \mathbf{d}u$$

where $\psi^1_{(1,1)}$, $\psi^1_{(2,2)}$ are the elements along the diagonal of the matrix exponential on

$$\psi^1 = \begin{bmatrix} \zeta + \frac{\Omega^1_A - \rho\tilde{\Omega}^1}{\sigma^2(1-\rho^2)} & \frac{\tilde{\Omega}^1 - \rho\Omega^1_A}{\sigma^2(1-\rho^2)} \\ \frac{\tilde{\Omega}^1 - \rho\Omega^1_B}{\sigma^2(1-\rho^2)} & \zeta + \frac{\Omega^1_B - \rho\tilde{\Omega}^1}{\sigma^2(1-\rho^2)} \end{bmatrix}$$

To arrive at Equation (7), I first present a more straightforward example. Consider the differential equation

$$\frac{\mathbf{d}X^A_t}{X^A_t} = \mu^A_t \mathbf{d}t + \sigma \left(\sqrt{\frac{1 + \sqrt{1-\rho^2}}{2}} \mathbf{d}Z^A_t + \sqrt{\frac{1 - \sqrt{1-\rho^2}}{2}} \mathbf{d}Z^B_t \right)$$

Its solution is

$$X^A_t = X_0 \exp \left[\int_0^t \left(\mu^A_u - \frac{1}{2}\sigma^2 \right) \mathbf{d}u + \int_0^t \sigma_p \mathbf{d}Z^1_t + \int_0^t \sigma_m \mathbf{d}Z^2_t \right]$$

Now taking the Malliavin derivative of X_T at date t and applying the chain rule of Malliavin calculus,

$$\mathscr{D}_t X_T^A = X_T^A \mathscr{D}_t X_0^A + X_T^A \left[\int_t^T \mathscr{D}_t \mu_u^A \mathbf{d}u + \sigma_p + \sigma_m \right]$$

which reduces to

$$\frac{\mathscr{D}_t X_T^A}{X_T^A} = \begin{bmatrix} \sigma_p & \sigma_m \end{bmatrix} + \frac{1}{\zeta}\left(1 - e^{-\zeta(T-t)}\right) \begin{bmatrix} \frac{\Omega^G - \rho\tilde{\Omega}^G}{\sigma^2(1-\rho^2)}\sigma_p + \frac{\tilde{\Omega}^G - \rho\Omega^G}{\sigma^2(1-\rho^2)}\sigma_m \\ \frac{\Omega^G - \rho\tilde{\Omega}^G}{\sigma^2(1-\rho^2)}\sigma_m + \frac{\tilde{\Omega}^G - \rho\Omega^G}{\sigma^2(1-\rho^2)}\sigma_p \\ \phi'\sigma_\mu \end{bmatrix}^\dagger$$

Applying the chain rule to the definition of $\omega_T(\eta_T^1, \eta_T^2)$ leads to (7) after using an integrating factor to find the solution to g_A^1 and g_B^1.

$\square$

B Additional Figures

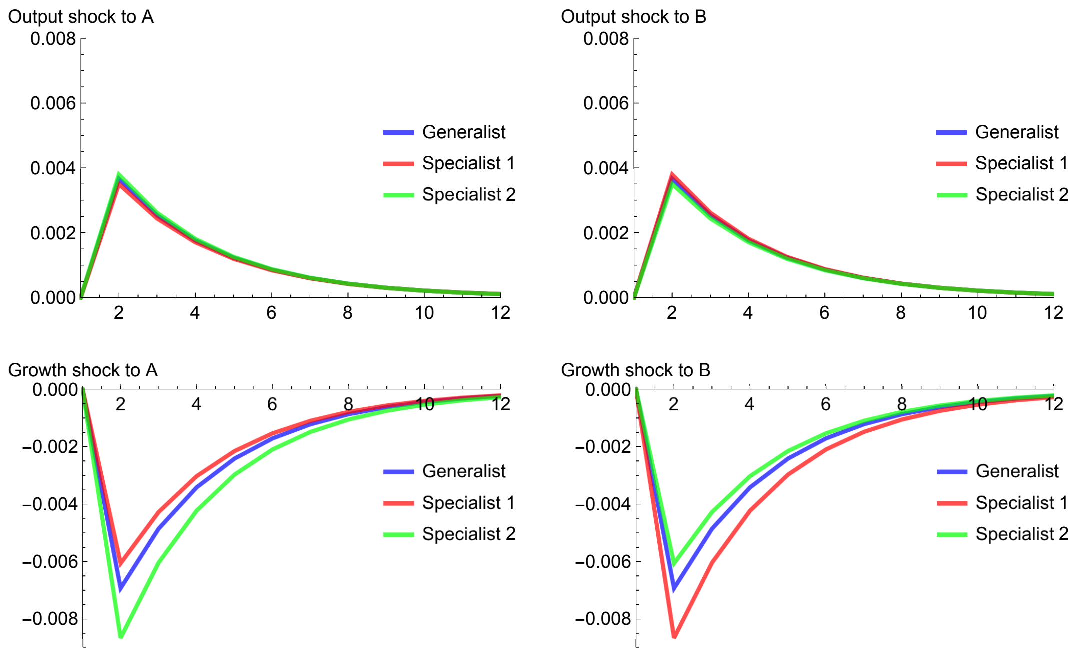

Figure B.1: **Impulse response of an agent's forecast error in response to a unit Brownian shock at $t = 1$ for $\phi' = 0.2,\ \phi = 0.3$.** Forecasts about μ^A (left), μ^B (right).

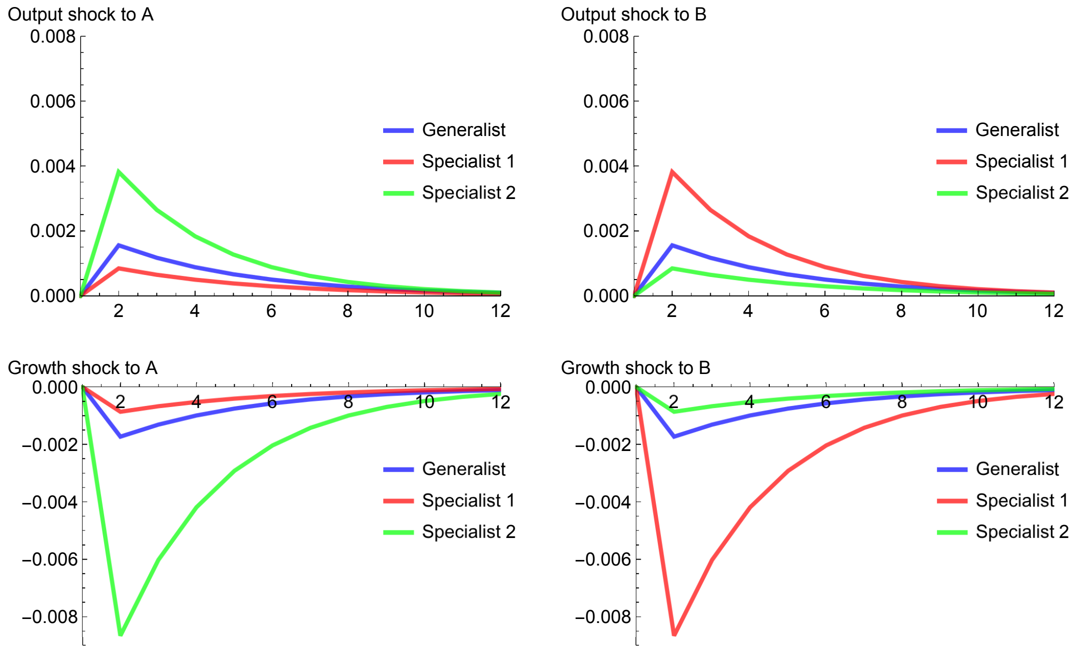

Figure B.2: **Impulse response of an agent's forecast error in response to a unit Brownian shock at $t = 1$ for $\phi' = 0.8,\ \phi = 0.9$.**

Figure B.3: **Dominance Regions and $\mathbb{E}^P\left[\omega_T\right]$ on the Edges.** The color corresponds to the range of values for $\mathbb{E}^P\left[\omega_T\right]$ for a given coordinate pair (ϕ', ϕ). The true informativeness of the signal is the midpoint of ϕ' and ϕ.